Chapter 17

SEQUENTIAL STATISTICAL DECISIONS MODEL, "BRIDE PROBLEM"

1. INTRODUCTION

Most of the decisions in a personal life and in organizations are made sequentially. That means that, at any given moment, one either makes the final decision, or postpones it hoping for better times. Statistical part of sequential decisions represents uncertainties of future events and a limited reliability of our observations.

The Bride problem is a good example of sequential statistical decisions (Wald, 1947; Wald, 1950). The dynamic programming is a conventional technique to optimize sequential decisions (Bellman, 1957). Applying the dynamic programming to specific problems, one develops specific algorithms, as usual. The algorithms, similar to these of bride's decision making, can be applied choosing the optimal time to buy durable goods, such as cars, houses, computers, to start new business, e.t.c..

Typical industrial applications of sequential decisions are the optimal stopping rules. For example, one can consider stopping of some nuclear plant as a marriage. One can evaluate stopping cost directly. Risks involved in running the plant, when one observes some deviations from the normal routine, can be considered as risks of waiting for the next proposal. That means that one hopes for something better while risking a bad outcome.

2. AVERAGE UTILITY

The Bride problem is to maximize the average utility of marriage by the optimal choice of a groom. Denote the actual goodness of a groom i by ω_i. Denote by s_i the bride's impression about the groom i. $p(\omega_i$ is a prior probability density of goodness ω_i. $p_s(s_i|\omega_i)$ is a probability density of impression s_i. Assume that goodness of different grooms are independent and identically distributed. This means that a prior probability density of goodness is

$$\begin{aligned} p(\omega_i, \omega_j) &= p(\omega_i)p(\omega_j), \\ p(\omega_i) &= p(\omega_j) = p(\omega). \end{aligned} \tag{17.1}$$

Suppose that impressions about the goodness of grooms are independent and identically distributed. Then the probability density of an impression s_i, given the goodness ω_i, is

$$\begin{aligned} p_s(s_i, s_j|\omega) &= p_s(s_i|\omega)p_s(s_j|\omega), \\ p_s(s_i|\omega) &= p_s(s_j|\omega) = p(s|\omega). \end{aligned} \tag{17.2}$$

Assume the Gaussian prior probability density of goodness

$$p(\omega) = \frac{1}{\sqrt{2\pi}\sigma_0} e^{-1/2(\frac{\omega-\alpha_0}{\sigma_0})^2}. \tag{17.3}$$

Here σ_0 is a prior variance and α_0 is a prior mean of goodness. For example, $\alpha_0 > 0$ shows an optimistic bride. $\alpha_0 < 0$ shows that a bride is pessimistic. Suppose, that a prior probability density of bride impressions is

$$p(s|\omega) = \frac{1}{\sqrt{2\pi}\sigma} e^{-1/2(\frac{s-\omega}{\sigma})^2}. \tag{17.4}$$

Here σ^2 is a variance of impressions around the true goodness ω.

Assume that both the groom goodness and the bride impression are random variables depending on many independent factors. This explains the Gaussian distributions (17.3) and (17.4). One defines a posterior probability density of goodness ω, given the impression s, by the Bayesian formula (Bayes, 1783)

$$p(\omega|s) = \frac{p(s|\omega)p(\omega)}{p_s(s)}. \tag{17.5}$$

Here

$$p_s(s) = \int_{-\infty}^{\infty} p(s|\omega)p(\omega)d\omega. \tag{17.6}$$

3. SINGLE-MARRIAGE CASE

Denote by d_i the bride's decision about the groom i

$$d_i = \begin{cases} 1, & \text{if } \textit{bride marry the groom } i, \\ 0, & \text{otherwise.} \end{cases} \quad (17.7)$$

Suppose that

$$\sum_{i=1}^{m} d_i = 1. \quad (17.8)$$

The last condition means that brides marry, and marry only once. Formally condition (17.8) defines the set of feasible decisions when the nth groom proposes

$$D_{N-n} = \begin{cases} 0 \textit{ and } 1, & \text{if } g_{N-n} = 0 \\ 0, & \text{if } g_{N-n} = 1, \end{cases} \quad (17.9)$$

Here g_{N-n} is the marriage index

$$g_{N-n} = 1 - \sum_{i=1}^{N-n-1} \quad (17.10)$$

The marriage index is zero, if the bride is marred. This prevents repeated marriages.

3.1 BELLMAN'S EQUATIONS

The expected utility function is $u(s)$. Here s is the impression made by the successful groom[1].

$$u(s) = \int_{-\infty}^{\infty} \omega p(\omega|s) d\omega. \quad (17.11)$$

Denote by $u_N(s)$ the expected utility function, if the impression of the last groom is s

$$u_N(s) = \int_{-\infty}^{\infty} \omega p(\omega|s) d\omega. \quad (17.12)$$

Comparing (17.11) and (17.12) one observes that

$$u(s) = u_N(s). \quad (17.13)$$

[1]By the "successful groom" we mean the groom that a bride marries.

This follows from independence assumptions (17.1) and (17.2). Denote by u_{N-1} the expected utility, if the impression of the $(N-1)$th groom is s and a bride is making the optimal decision $d = d_{N-1}(s) \in D_{N-1}$

$$u_{N-1}(s) = \max_d (du(s) + (1-d)u_N), \tag{17.14}$$

$$d_{N-1}(s) = arg \max_d (du(s) + (1-d)u_N). \tag{17.15}$$

Here

$$u_N = \int_{-\infty}^{\infty} u_N(s) p_s(s) ds. \tag{17.16}$$

Following the same pattern, we define the expected utility, if the impression of the $(N-n)$th groom is s and the bride is making the optimal decision $d = d_{N-n}(s) \in D_{N-n}$

$$u_{N-n}(s) = \max_d (du(s) + (1-d)u_{N-n+1}), \tag{17.17}$$

$$d_{N-n}(s) = arg \max_d (du(s) + (1-d)u_{N-n+1}). \tag{17.18}$$

Here

$$u_{N-n+1} = \int_{-\infty}^{\infty} u_{N-n+1}(s) p_s(s) ds. \tag{17.19}$$

Note, that the utility $u(s)$ of accepting a proposal in expressions (17.17) and (17.18) is a function only of impression s. It does not depend on the proposal number $N-n$. That follows from independence assumptions (17.1) and (17.2). Solving these recurrent equations one defines the sequence of optimal decision functions $d_{N-n}(s) \in D_{N-n}$ and the expected utilities $u_{N-n}(s)$. This should be done for all possible impressions $s \in (-\infty, \infty)$ and for all numbers $n = 1, ..., N-1$. One cannot do that in continuous case. Therefore, one uses a discrete approximation

3.2 DISCRETE APPROXIMATION

From expressions (17.11) and (17.12), replacing the integrals by sums one obtains that

$$u_N(s) = u(s) = 2M/K \sum_{k=1}^{K} \omega_k p(\omega_k|s). \tag{17.20}$$

From expression (17.16)

$$u_N = 2M/K \sum_{k=1}^{K} u_N(s_k) p_s(s_k). \tag{17.21}$$

From expression (17.19)

$$u_{N-n+1} = 2M/K \sum_{k=1}^{K} u_{N-n+1}(s_k)p_s(s_k). \tag{17.22}$$

Here $\omega_k \in [-M, M]$, $\omega_1 = -M$, $\omega_K = M$ and $s_k \in [-M, M]$, $s_1 = -M$, $s_K = M$. That is a discrete approximation of the recurrent equations. All possible impressions $s_k \in [-M, M]$ and all numbers $n = 1, ..., N-1$ are considered. One sets the number of iterations K by the accuracy needed.

The results are sequences of optimal decision functions $d_{N-n}(s_k)$ and the expected utilities $u_{N-n}(s_k)$. These sequences are stored in a set of arrays which define how the optimal decisions d and the expected utilities u depend on the possible impressions s_k, $k = 1, ..., K$. Doing this, one avoids the repeated calculations. Large arrays is a disadvantage of this procedure.

3.3 INCLUDING THE WAITING COST

The waiting losses are important in many real-life sequential decision problems. Denote by c the loss of waiting for the next groom. Including this parameter into Bellman equations one obtains

$$u_{N-1}(s) = \max_d (du_N(s) + (1-d)(u_N - c)), \tag{17.23}$$

$$d_{N-1}(s) = arg \max_d (du_N(s) + (1-d)(u_N - c)). \tag{17.24}$$

In a similar way one defines the expected utility if the impression of the $(N-n)$-th groom is s and the bride is making the optimal decision $d_{N-n}(s)$

$$u_{N-n}(s) = \max_d (du_N(s) + (1-d)(u_{N-n+1} - c)), \tag{17.25}$$

$$d_{N-n}(s) = arg \max_d (du_N(s) + (1-d)(u_{N-n+1} - c)). \tag{17.26}$$

The other expressions remains the same.

3.4 NON-LINEAR CASE

Expression (17.12) was defined assuming the linear bride's utility function. It was supposed that bride's utility is equal to the goodness of groom $u(\omega) = \omega$. In the real life, utility functions are nonlinear (see chapter 13) and non-convex, as usual. Then expression (17.12) is replaced by this integral

$$u_N(s) = \int_{-\infty}^{\infty} u(\omega) p(\omega|s) d\omega. \tag{17.27}$$

4. MULTI-MARRIAGE CASE

Condition (17.8) implies that brides marry and marry only once. No divorce is permitted. That is wrong, if the "marriage" represents buying a Personal Computer (PC). We illustrate this by a simple "Buy-a-PC" example.

4.1 " BUY-A-PC" EXAMPLE

Define PC parameters by a vector $g = (g_1, g_2, g_3)$. Here g_1 is the speed of CPU in MHz, g_2 is the volume of RAM in MB, and g_3 is the volume of HD in GB. Express a subjective utility of PC by the weighted sum $\omega = a_1 g_1 + a_2 g_2 + a_3 g_3$. Here a_i are expressed in \$ per unit.

Assume that a prior probability density of PC utilities is Gaussian

$$p_t(\omega) = \frac{1}{\sqrt{2\pi}\sigma_0} e^{-1/2(\frac{\omega - \alpha_t}{\sigma_0})^2}. \tag{17.28}$$

Here σ_0 is a prior variance and α_t is a prior mean of PC utilities. Suppose that $\sigma_0 = constant$ and that $\alpha_t = \alpha_0 + \alpha_1 t$. This means that expected PC utility is increasing linearly. The expected diversity remains the same.

The PC price in \$ is denoted by l. Suppose that the price of PC depends linearly on the weighted sum

$$l = b_1 g_1 + b_2 g_2 + b_3 g_3. \tag{17.29}$$

Here parameters b_i are defined in \$ per unit and reflect the market prices of CPU, RAM, and HD. Expressing the price l as a function of utility ω:

$$l = h_0 \omega. \tag{17.30}$$

Here

$$h_0 = \frac{b_1 g_1 + b_2 g_2 + b_3 g_3}{a_1 g_1 + a_2 g_2 + a_3 g_3}. \tag{17.31}$$

Assume that one observes the utility ω exactly. This means that the impression $s = \omega$ and that impression errors $\sigma = 0$ in expression (17.2). That simplifies the problem.

4.2 BELLMAN'S EQUATIONS

The expected utility function is $u(\omega, q)$. Here ω is the utility of a new PC. q is the utility of the old PC, to be replaced by the new one. Consider a "horizon" of N years. During a year one can change PC only once, if one wishes.

Denote by $u_N(\omega, q)$ the maximal expected utility in the year N

$$u_N(\omega, q) = \max_d (d\omega + (1-d)(q_N - c_N)). \tag{17.32}$$

There are two possible decisions $d = \{0, 1\}$. The decision $d = 1$ means to buy a new PC. The utility of the new PC is ω. The utility of the old one is q.

The utility of the decision $d = 0$, to keep the old PC in the last year N, is $q_N + c_N$. Here $c_N = \tau_N - l_N$ is the price of refusing to buy a new PC. This includes the waiting losses τ minus the price l_N of new PC in the year N defined by (17.30). It is assumed that we abandon the old PC as soon as we obtain the new one. Therefore, one "wins" the price l of the new PC by using the old PC. The optimal decision in the last year N

$$d_N^* = \begin{cases} 1, & \text{if } \omega_N > q_N - c_N, \\ 0, & \text{if } \omega_N \leq q_N - c_N. \end{cases} \tag{17.33}$$

Denote by $u_{N-1}(\omega, q)$ the maximal expected utility in the year $N-1$.

$$u_{N-1}(\omega, q) = \max_d (d\omega + (1-d)(u_N(q_N) - c_{N-1})). \tag{17.34}$$

Here $u_N(q)$ is the maximal expected utility in the year N, if the utility of the old PC is q.

$$u_N(q) = \int_{-\infty}^{\infty} u_N(\omega, q) p_N(\omega) d\omega. \tag{17.35}$$

$p_N(\omega)$ is a prior probability density of ω defined by expression (17.28) at a time $t = N$.

$$q_N \quad = \quad \begin{cases} \omega_{N-1}, & \text{if } q_N < q^*_{N-1}, \\ q_{N-1}, & \text{if } q_N \geq q^*_{N-1}. \end{cases} \tag{17.36}$$

Here q^*_{N-1} is obtained from this equation

$$\omega_{N-1} = u_N(q^*_{N-1}) - c_{N-1} \tag{17.37}$$

The optimal decision in the year $N-1$

$$d^*_{N-1} \quad = \quad \begin{cases} 1, & \text{if } \omega_{N-1} > u_N(q_N) - c_{N-1}, \\ 0, & \text{if } \omega_{N-1} \leq u_N(q_N) - c_{N-1}. \end{cases} \tag{17.38}$$

Denote by $u_{N-i}(\omega, q)$ the maximal expected utility in the year $N-i$, $i = 1, ..., N-1$. Then

$$u_{N-i}(\omega, q) = \max_d (d\omega + (1-d)(u_{N-i+1}(q_{N-i+1}) - c_{N-i})). \tag{17.39}$$

Here $u_{N-i+1}(q)$ is the maximal expected utility in the year $N-i+1$, if the utility of the old PC is q.

$$u_{N-i+1}(q) = \int_{-\infty}^{\infty} u_{N-i+1}(\omega, q) p_{N-i+1}(\omega) d\omega. \tag{17.40}$$

$p_{N-i+1}(\omega)$ is a prior probability density of ω defined by expression (17.28) at a time $t = N-i+1$.

$$q_{N-i+1} \quad = \quad \begin{cases} \omega_{N-i}, & \text{if } q_{N-i+1} < q^*_{N-i}, \\ q_{N-i}, & \text{if } q_{N-i+1} \geq q^*_{N-i}. \end{cases} \tag{17.41}$$

q^*_{N-i} is obtained from the equation

$$\omega_{N-i} = u_{N-i+1}(q^*_{N-i}) - c_{N-i} \tag{17.42}$$

The optimal decision in the year $N-i$

$$d^*_{N-i} \quad = \quad \begin{cases} 1, & \text{if } \omega_{N-i} > u_{N-i+1}(q_N - i + 1) - c_{N-i}, \\ 0, & \text{if } \omega_{N-i} \leq u_{N-i+1}(q_{N-i+1}) - c_{N-i}. \end{cases} \tag{17.43}$$

Here the decision functions d^*_{N-i} depend on two variables ω, q defining the corresponding utilities of the new and the old computer. That is difficult for both the calculations and grafical representations.

One applies the discrete approximation such as in Section 3.2. The optimal decision functions depend on two variables ω and q. Therefore, one needs K times more calculations to obtain the same accuracy as in the single marriage case (see Section 3.2).

To solve real life problems the "directly unobservable" factors should be included into the "Buy-a-PC" model. For example , one can estimate the expected life-span $T(s)$ of PC, given the impression s, by the Bayesian formula

$$T = \int_{-\infty}^{\infty} \tau p(\tau|s) d\tau, \tag{17.44}$$

$$p(\tau|s) = \frac{p(s|\tau)p(\tau)}{p_s(s)}. \tag{17.45}$$

Here τ is the life-span of PC and s is the user's impression about its reliability. The impression s depends on the warranty, the reputation of the manufacturer and on some subjective factors, too.

ONE-STEP-AHEAD APPROXIMATION

To simplify the calculations, the "One-Step-Ahead" approximation is suggested. Here one assumes that the next step is the last one and introduces the parameter $r = \omega_{N-1} - q_{N-1} + c_{N-1}$.

The one-step-optimal decision

$$d_{N-1}(r) \quad = \quad \begin{cases} 1, & \text{if } r \geq 0, \\ 0, & \text{if } r < 0. \end{cases} \tag{17.46}$$

Following the same pattern, one defines the maximal expected utility in $(N-n)$th year

$$u_{N-n}(\omega, q) = \max_d (d\omega_{N-n} + (1-d)u_{N-n+1}(q)), \tag{17.47}$$

where

$$u_{N-n+1}(q) = \int_{-\infty}^{\infty} u_{N-n+1}(\omega, q) p_{N-n+1}(\omega) d\omega. \tag{17.48}$$

Here

$$q \quad = \quad \begin{cases} \omega_{N-1}, & \text{if } r \geq 0, \\ q_{N-1}, & \text{if } r < 0. \end{cases} \tag{17.49}$$

The one-step-optimal decision

$$d_{N-n}(r) \quad = \quad \begin{cases} 1, & \text{if } r \geq 0, \\ 0, & \text{if } r < 0. \end{cases} \tag{17.50}$$

The one-step-optimal decision $d_n(r)$ is a function of difference $r = \omega_{N-n} - u_{N-n+1}(q)$ that depends on two parameters ω and q. Therefore, one defines it as a function of critical pairs (ω, q) depending on time $t = n$.

Solving these recurrent equations one defines the sequence of one-step-optimal decision functions $d_{N-n}(r)$ and the expected utilities $u_{N-n}(\omega, q)$. This should be done for all possible values of ω and q.

The One-Step-Ahead approximation, assuming that $r = \omega_{N-n} - q_{N-n} - l_{N-n}$, reduces the computing time to a level of single marriage case. However, one needs to estimate the approximation error.

5. SOFTWARE EXAMPLES

Two interactive versions of the Bride problem are implemented as Java applets. These, and the corresponding $C++$ software are on websites (see section 4.).
Figure 17.1 shows the input window of the Bride problem (Kajokas, 1997)
Figure 17.2 shows simulated results of the "optimal marriage".
Figure 17.3 shows the optimal decision function defining how the critical "goodness" of a groom depends on his number.
Figure 17.4 shows the help window of the "Buy-a-Car" version of the Bride problem (Mikenas, 1998).
Figure 17.5 shows the input window of the "Buy-a-Car" problem.
Figure 17.6 shows the output window of the "Buy-a-Car" problem.
Figure 17.7 shows the optimal decision function of the "Buy-a-Car" problem.

The "Buy-a-Car" algorithm is just a first approximation. It should be improved. For example, improving the By-a-PC model one should

- remove the "single marriage" condition (17.8),
- add "Change-a-Car" cost,
- express desirable car properties (see Figure 17.5), using conditions of Pareto optimality (Mockus, 1989b),
- extend the property list by including properties that are not directly observable, such as the reliability of a car,